PROCEEDINGS OF IAM, V.7, N.2, 2018, pp.194-215

APPLICATION OF FINITE INTEGRAL TRANSFORMATION METHOD TO THE SOLUTION OF MIXED PROBLEMS FOR PARABOLIC EQUATIONS WITH A CONTROL*

Elmaga A. Gasymov
Baku State University, Az 1141, Baku, Azerbaijan
gasymov-elmagha@rambler.ru

Abstract. In the present paper, we use the method of finite integral transformation to the solution of mixed problems for parabolic equations with a control and with irregular boundary conditions. The analytic representation of the solution of the considered problem is obtained. For the accessibility of a wide range of readers on some (parabolic) model problems, the method of finite integral transformation it self is stated.

Keywords: Classic solution, finite integral transformation method, control equation, irregular boyndary condition.

AMS Subject Classification: 35K10, 35K35.

1. Introduction

Throughout the theoretical and applied importance, a mixed problem for partial differential equations refer to one of the urgent problems of mathematics and mathematical physics. Some problems of electrodynamics, problems of underground hydromechanics, nonstationary problems for mathematical physics equations are reduced to such problems. Symbolic calculus used by engineer-electrician O. Hevyside was one of the convenient, but mathematically not reasonable tools.

In the beginning of the XIX century, Fourier suggested the method of separation of variables for integration of some linear partial differential equations under the given boundary and initial conditions. Application of the Fourier method to the solution of mixed problems with separated variables reduces to the problem of expansion of an arbitrary function from some class in eigen functions corresponding to the spectral problem.

In 1827, for solving mixed problems with constant coefficients, Cauchy [2] suggested the residue method the essence of which is in representation of an "arbitrary" function in the form of integral residue. One of the methods for solving mixed problems for partial differential equations is the method of integral transformations that was successfully used by Laplace , A.L. Cauchy [2], M.L. Rasulov [10] and others.

In his researches M.L. Rasulov used the integral transformation

$$\overline{\varphi}(\lambda)=\int_0^\infty e^{-\lambda^2 t}\varphi(t)dt, \tag{1}$$

* Работа была представлена на семинаре Института Прикладной Математики 18.09.2018

and for $\overline{\varphi}(\lambda)$ assumed that $\overline{\varphi}(\lambda)$ is an analytic function in the domain

$$R_\delta = \left\{ \lambda : |\lambda| \geq R,\ |\arg \lambda| \leq \frac{\pi}{4} + \delta, \delta > 0 \right\},$$

tending to zero as $|\lambda| \to \infty$ uniformly with respect to $\arg \lambda$ (see the conditions of theorem 1.1 on page 152 of the paper [10]).

Note that if for $\lambda \in R_\delta$ satisfying the inequality, $\operatorname{Re}(-\lambda^2) > 0$, M.L. Rasulov determines $\overline{\varphi}(\lambda)$ by formula (1), then this integral, generally speaking, diverges. But if $\lambda \in \Omega = \left\{ |\lambda| \geq R,\ |\arg \lambda| \leq \frac{\pi}{4} \right\}$ Rasulov would take $\overline{\varphi}(\lambda)$ in the form of (1) and in domain $R_\delta \setminus \Omega$ determined the function $\overline{\varphi}(\lambda)$ by analytic continuation, then the conditions of existence of such a continuation should be clarified additionally so that this assumption be fulfilled. Furthermore , M.L. Rasulov, except one special function $\varphi(t) = \frac{1}{\sqrt{t}}$ (from p.245, [10]) does not show the class of functions $\varphi(t)$ for which $\overline{\varphi}(\lambda)$ satisfies this assumption.

In [10], [11] it is assumed that under suitable numberings $\theta_j(x)$ (θ_j are the roots of the characteristic equation of spectral problem), the following conditions are fulfilled

$$\operatorname{Re}[\lambda\theta_1(x)] \leq \operatorname{Re}[\lambda\theta_2(x)] \leq \ldots \leq \operatorname{Re}[\lambda\theta_n(x)],\ x \in [\alpha, \beta],\quad \lambda \in \omega, \tag{2}$$

where ω is some infinite part of λ plane, wherein we look for asymptotic behavior of the system of fundamental particular solutions of a homogeneous equation corresponding to spectral problem. In applications, feasibility of condition (2) reduces to the fact that:

A) arguments $\theta_j(x)$ and arguments of their differences are independent of $x \in [\alpha, \beta]$ (for example, see: restraint 3^0 on page 23, [10]), that in our opinion is a more rigid restraint.

Note that when solving mixed problems [4] we are restricted in consideration of a parametric problem not on the whole infinite part of λ - plane but only in some part (in the sector $|\arg \lambda| \leq \frac{\pi}{4} + \delta$) of λ-plane and because of that we get rid of rigid constraint *A*).

There exists a wide range of mixed problems of theoretical and practical importance that are not solved by the known Fourier, G.D. Birkhoff [1], Ya. D. Tamarkin [11], M.A. Naimark [9], M.L. Rasulov [10] methods. In the case of irregular boundary conditions, expansion in eigen and associated functions has a number of specific features.

The authors resecrch shows that when solving these problems it is not obligatory to use Brikhoff- Tamarkin-Naimark-Rasulov's expansion formulas.

In this paper we suggest the method of finite integral transformation that admits to find the solution of irregular mixed problems under more general boundary conditions and weaker constraints on the problem data.

1.1. Finite integral transformation

Let $f(t)$ be a complex, $\omega(t)$ a real function of the real argument t $(0 \le t \le T,$ T is some positive number) and $f,\ \omega,\ f\cdot\omega \in L([0,T])$.

Definition. We call the function $\tilde{f}_{\pm}(\lambda,t)$, determined by the formula

$$\begin{aligned} \tilde{f}_{-}(\lambda,t) &= \int_0^t \omega(\tau)\exp\left[-\lambda\int_0^\tau \omega(\eta)d\eta\right] f(\tau)d\tau, \\ \tilde{f}_{+}(\lambda,t) &= \int_t^T \omega(\tau)\exp\left[-\lambda\int_0^\tau \omega(\eta)d\eta\right] f(\tau)d\tau,\quad t\in[0,T], \end{aligned} \tag{3}$$

where λ is a complex number, the image of the function $f(t)$

We have:

Theorem E. Let $\omega(t)\in C((0,T])\cap L([0,T])$, $\int_\tau^t \omega(\eta)d\eta > 0$ for $0\le\tau<t\le T$, $f(t)$ be bounded and continuous (except denumerable number of points at which it may have discontinuity of first kind) with respect to} $t\in[0,T]$.

Then for all $t(0<t<T)$, the function $f(t\pm 0)\equiv \lim\limits_{\tau\to t\pm 0} f(\tau)$ is represented by its own image in the form

$$f(t\pm 0) = \frac{1}{(\pi-2\theta)\sqrt{-1}}\int_L \exp\left[\lambda\int_0^t \omega(\tau)d\tau\right]\tilde{f}_{\pm}(\lambda,t)\,d\lambda, \tag{4}$$

where L is an in finite smooth line in λ -plane whose a rather distant part coincides with continuation of the rays $\arg(\lambda+a)=\pm\left(\frac{\pi}{2}+\theta\right)$; a,θ $(0<\theta<\pi/2)$ are some constants, and in (4) the integral with respect to L is understood in the sense of principal value.

Assume $f_i(t)=f(t)$ for t$t_i<t<t_{i+1}$, $f_i(t_i)=f(t_i+0)$, $f_i(t_{i+1})=f(t_{i+1}-0)$, where $t_i(0=t_0<t_1<\ldots<t_m=T)$ are some points.

Corollary. If $\omega(t)\in C([0,T])$, $\omega(t)>0$ for $t\in[0,T]$ (t)>0 and the functions $f_i(t)$ are absolutely continuous with respect to $t\in[t_i, t_{i+1}]$, $(i=0,...,m-1)$ (*m* is a natural number), then (4) holds for $\theta=0$ as well.

1.2. Finite integral transformation method(for parabolic equations)

Here for a wide range of readers, on the following irregular model problem 1 for parabolic equations we state the finite integral transformation method.

Model problem 1. To find the classic solution $u=u(x,t)$ of the equation

$$\frac{\partial u}{\partial t}=a^2\frac{\partial^2 u}{\partial x^2}+F(x,t)\quad 0<x<1,\quad 0<t\le T\le\infty \tag{5}$$

satisfying the irregular conditions

$$\left.\frac{\partial u}{\partial t}\right|_{x=0}=\varphi_1(t),\quad \int_0^1 K(x)u(x,t)dx=\varphi_2(t)\quad 0<t\le T, \tag{6}$$

and the initial condition

$$u(x,t)\big|_{t=0}=f(x),\quad x\in(0,1), \tag{7}$$

where F,φ_1,φ_2, f are the known functions, a-*const*

1^0. Let equation (1) be parabolic in I.G. Petrovsky sense, i.e. let

$$a=|a|e^{\sqrt{-1}\arg a},\quad |\arg a|\le\frac{\pi}{4}-\theta, |a|>0,$$

where θ is some number satisfying the inequality

$$0<\theta<\frac{\pi}{4}. \tag{8}$$

2^0. Let the functions $F(x,t),\varphi_1(t),\varphi_2(t), f(x)$ be continuous for $0\le x\le 1$, $0\le t\le T$.

Stage 1. Obtaining operation problem

For $t>0$ applying the finite integral transformation

$$\tilde{\phi}(t,\lambda)=\int_0^t e^{-\lambda^2\tau}\phi(\tau)d\tau, \tag{9}$$

(λ is a complex parameter) to (5)-(6) and using (7) and then performing on the image $\tilde{u}(x,t,\lambda)$ operations corresponding to the given operations on $u(x,t)$, we get the following operation problem (10),(11):

$$\left(a^2\frac{\partial^2}{\partial x^2}-\lambda^2\right)\tilde{u}(x,t,\lambda)=\exp(-\lambda^2 t)u(x,t)-f(x)-\tilde{F}(x,t,\lambda),\quad 0<x<1, \tag{10}$$

$$\tilde{u}(x,t,\lambda)\big|_{x=0} = -\frac{1}{\lambda^2}e^{-\lambda^2 t}v(t) + \frac{1}{\lambda^2}(f(0) + \tilde{\varphi}_1(t,\lambda));$$

$$\int_0^1 K(x)\tilde{u}(x,t,\lambda)dx = \tilde{\varphi}_2(t,\lambda), \tag{11}$$

where

$$\tilde{u}(x,t,\lambda) = \int_0^t e^{-\lambda^2\tau}u(x,\tau)d\tau,\ \ \tilde{F}(x,t,\lambda) = \int_0^t e^{-\lambda^2\tau}F(x,\tau)d\tau,$$

$$v(t) \equiv u(x,t)\big|_{x=0},\ \tilde{\varphi}_k(t,\lambda) = \int_0^t e^{-\lambda^2\tau}\varphi_k(\tau)d\tau. \tag{12}$$

Remark 1. One our distinction from the authors engaged in such problems is that unlike the operation problem constructed by us in the present case the right hand side of the operation problem contains the sought-for function $u(x,t)$ as well.

Stage 2. Studying the parametric problem.

For solving the operation problem (10)-(11) at first we study the parametric problem (13)-(14) corresponding to it

$$a^2 y'' - \lambda^2 y = \psi(x), \quad x \in (0,1) \tag{13}$$

$$U_1(y) \equiv y/_{x=0} = \gamma_1,\ \ U_2(y) \equiv \int_0^1 K(x)y(x,\lambda)dx = \gamma_2, \tag{14}$$

where $\psi(x) \in C([0,1]), \gamma_1, \gamma_2$ are arbitrary numbers.

In [4] we have shown that when studying mixed problems for parabolic systems of order 2*p* it suffices to consider the domain of change of parameters λ in the form

$$R_\delta = \left\{\lambda : |\lambda| \geq R, \quad |\arg\lambda| \leq \frac{\pi}{4p} + \delta, \delta > 0\right\}.$$

Consequently, here and in the sequel, we will assume that

$$\lambda \in R_\delta = \left\{\lambda : |\lambda| \geq R, \quad |\arg\lambda| \leq \frac{\pi}{4p} + \delta\right\},$$

where *R* is a rather large positive number (see remark 5),

$$\delta (0 < \delta < \theta) \tag{15}$$

is an arbitrarily fixed number (see remarks 4 and 7).

We take the system of fundamental particular solutions of a homogeneous equation corresponding to (13) in the form

$$y_1(x,\lambda)=\exp\left(-\frac{\lambda}{a}x\right),\quad x\in[0,1],\ y_2(x,\lambda)=\exp\left(-\frac{\lambda}{a}(1-x)\right), \tag{16}$$

the fundamental solution of equation (13) we take in the form

$$P(x,\xi,\lambda)=-\frac{1}{2a\pi}\exp\left(-\frac{\lambda}{a}|x-\xi|\right). \tag{17}$$

Note that here fundamental solution $P(x,\xi,\lambda)$ and the systems of fundamental particular solution $y_1(x,\lambda),\ y_2(x,\lambda)$, are chosen so that for $\lambda\in R_\delta, |\lambda|\to\infty$ they are decreasing functions with respect to λ.

Remark 2. It is known that the Green functions of the problem (13)-(14) are independent on the choice of the fundamental solution $P(x,\xi,\lambda)$ and the system of fundamental particular solutions $y_1(x,\lambda),\ y_2(x,\lambda)$. Here their such choice for $\lambda\in R_\delta$ frees us from the operations performed in [10, 11], over the determinants contained in the expression of the Green function.

The denominator of the Green function of the problem (13)-(14) will be

$$\Delta(\lambda)=\begin{vmatrix} U_1(y_1) & U_1(y_2) \\ U_2(y_1) & U_2(y_2)\end{vmatrix}. \tag{18}$$

Expanding the determinant (18), we have

$$\Delta(\lambda)=\alpha_M\lambda^M+\alpha_{M-1}\lambda^{M-1}+\ldots+\alpha_{M-S}\lambda^{M-S}+O(\lambda^{M-S-1}),\lambda\in R_\delta, \tag{19}$$

where M is the highest possible degree with respect to λ, S is some nonnegative integer, α_ν are some numbers.

Incidentally we note that we can take the number S contained in (19) rather large (i.e. for $\lambda\in R_\delta$ for $\Delta(\lambda)$ we can get more exact asymptotics) if the functions contained in the left hand side of the parametric problem (13), (14) are rather smooth.

Remark 3. In [4] for the equations with variable coefficients and "general" , boundary conditions sufficient conditions imposed on the coefficients of the parametric problem that provide to obtain asymptotics $\Delta(\lambda)$ to the required accuracy S for $\lambda\in R_\delta$ are given.

Definition 1. We say that boundary conditions of parametric problem (13)-(14) are well-imposed if at least one of the numbers

$$\alpha_M, \alpha_{M-1}, \ldots, \alpha_{M-S}\ (\mathit{from}\ (19)) \tag{20}$$

is nonzero.

From this definition we immediately get:

-If the boundary conditions of the parametric problem are regular in the sense of Birkhoff -Tamarkin - Naimark-Rasulov, then they are well-posed by our definition.

But the inverse statement is not true

3^0 . Let irregular conditions (14) be well-posed and in the sequence (20) the first nonzero number be α_q , where $q(q \in Z)$ is some integer.

We chose the number R so large that

$$|\Delta(\lambda)| \geq \frac{1}{2}|\alpha_q||\lambda|^q \quad for \quad \lambda \in R_\delta \tag{21}$$

Now show the sufficiency of the condition providing well-posedness of irregular conditions (14).

4^0. Let $K(x) \in C^n([0,1]),\ K^{(n-1)}(1) \neq 0,\ K^{(j)}(1) = 0$ for $j \leq n-2$, where n is some natural number.

Taking into account (16) in (18), we get

$$\Delta(\lambda) = \int_0^1 K(x) e^{-\frac{\lambda}{a}(1-x)} dx - e^{-\frac{\lambda}{a}} \int_0^1 K(x) e^{-\frac{\lambda}{a}x} dx. \tag{22}$$

Using the constraint 4^0 , integrating the following integral by parts, we have

$$\int_0^1 K(x) e^{-\frac{\lambda}{a}(1-x)} dx = -\left(-\frac{a}{\lambda}\right)^n K^{(n-1)}(1) + e^{-\frac{\lambda}{a}} \sum_{j=1}^{n} \left(-\frac{a}{\lambda}\right)^j K^{(j-1)}(0) +$$
$$+\left(-\frac{a}{\lambda}\right)^n \int_0^1 K^{(n)}(x) e^{-\frac{\lambda}{a}(1-x)} dx. \tag{23}$$

Taking into account 1^0 we have

$$\left| e^{-\frac{\lambda}{a}} \right| \leq e^{-\varepsilon\left|\frac{\lambda}{a}\right|}, \quad as \quad \lambda \in R_\delta, \tag{24}$$

where $\varepsilon = \sin(\theta - \delta) > 0.$

Remark 4. In (15) selection of the number δ satisfying the inequality $\delta < \theta$ provides positivity of the number ε contained in (24).

Note that

$$\xi^p e^{-\xi} \leq c_p, \quad \xi \in [1, \infty), \quad \lim_{\varepsilon \to \infty} \xi^p e^{-\xi} = 0, \tag{25}$$

where p is any real number, c_p is some positive constant.

Taking into account (23) (24), (25) in (22) , we get

$$\Delta(\lambda) = \frac{1}{\lambda^n}(-1)^{n+1} a^n K^{(n-1)}(1) + O\left(\frac{1}{\lambda^{n+1}}\right), \quad \lambda \in R_\delta \tag{26}$$

that shows well -posedness of irregular conditions (14).

Thus we established

Lemma 1. *Let constaints 1^0, 4^0 be fulfilled. Then for* $\lambda \in R_\delta$ *for the equation (13) the irregular conditions (14) are well-posed.*

Now we choose the number *R* so large that

$$\left|\Delta(\lambda)\right| \geq \frac{1}{\left|\lambda\right|^n}\frac{1}{2}\left|a\right|^n\left|K^{(n-1)}(1)\right| \quad as \quad \lambda \in R_\delta . \tag{27}$$

Remark 5. The numbers *R* is chosen from the following two conditions:

-in the domain R_δ to find to the required accuracy the asymptotic representation of the system of fundamental particular solutions of homogeneous equation corresponding to the equations of the parametric problem (in the present model example we did not need it)

-in the domain R_δ for $\Delta(\lambda)$ -denominator of the Green function of the parametric problem to get the lower estimate of type (21) (in the presen case of type (27)).

From (27) it follows that $\Delta(\lambda) \neq 0$ for $\lambda \in R_\delta$, this inequality implies the validity of the following classic theorem.

Theorem 1. *Let the constraints' 1^0, 4^0 be fulfilled,* $\psi(x) \in C([0,1])$, γ_1 *and* γ_2 *be arbitrary numbers, then for* $\lambda \in R_\delta$ *the parametric problem (13,14)*

-has a unique solution,

-this solution is represented by the formula [4]

$$y(x,\lambda) = \delta(x,\lambda,\gamma_1,\gamma_2) + \int_0^1 G(x,\xi,\lambda)\psi(\xi)d\xi, \quad x \in (0,1) \tag{28}$$

$$G(x,\xi,\lambda) = P(x,\xi,\lambda) + G_1(x,\xi,\lambda)$$

The function $\delta(x,\lambda,\gamma_1,\gamma_2)$ with respect to γ_1 and γ_2 is linear , i.e.

$$\delta(x,\lambda,\gamma_1 + q\beta_1\gamma_2 + q\beta_2) = \delta(x,\lambda,\gamma_1,\gamma_2) + q\delta(x,\lambda,\beta_1,\beta_2). \tag{29}$$

Taking into account (24) in (16) for $\lambda \in R_\delta$ we get the validity of the following inequalities

$$\left|y_k(x,\lambda)\right| \leq \rho(\left|\lambda\right|,x), \quad x \in [0,1], (k = 1,2), \tag{30}$$

where

$$\rho(\left|\lambda\right|,x) = \exp\left(-\varepsilon\left|\frac{\lambda}{a}\right|x\right) + \exp\left(-\varepsilon\left|\frac{\lambda}{a}\right|(1-x)\right).$$

Stage 3. Inversion formulas.

At this stage we get some inversion formulas connected with:

$\delta(x,\lambda,\gamma_1,\gamma_2)$ is the solution of homogeneous inversion formulas equation corresponding to (13), satisfying inhomogeneous irregular conditions (14),
- $G(x,\xi,\lambda)$ is the Green function of the parametric problem (13), (14).

Let L be an infinite open smooth line in the domain R_δ whose rather distant part coincides with the continuation of the rays $\arg\lambda = \pm\left(\frac{\pi}{4}+\delta\right)$. In the sequel, let $R < R_1 < R_2 < \ldots$, and $\lim_{m\to\infty} R_m = \infty$ L_m be a part of L remaining interior to the circle of radius R_m; and C_m be a part of a circle of radius R_m; (centered at the origin of coordinates of λ - plane) remaining in the domain R_δ. We have

Lemma 2. *Let $g(\lambda)$ be an analytic function with respect to $\lambda \in R_\delta$ and $|g(\lambda)| \le const|\lambda|^S$ for $\lambda \in R_\delta$ where $s,(s\in Z)$ is some integer. Then for $0 < x < 1$, we have the following formula of inversion to zero*

$$\int_L g(\lambda) y_k(x,\lambda)d\lambda = 0, \quad k = 1,2$$

where y_k is from (16). Here and in the sequel, the integral with respect to L is understood in the sense of principal value.

Lemma 3. *Under constraints 1^0 and 4^0 for any numbers γ_1 , γ_2 and for any integer $s,(s\in Z)$ we have the following formula of inversion to zero*

$$\int_L \lambda^s \delta(x,\lambda,\gamma_1,\gamma_2)d\lambda = 0, \quad 0 < x < 1 \quad . \tag{31}$$

Lemma 4. *Under constraints 1^0 and 4^0 if $\psi(x)$ is a piece-wise continuous function in [0,1] then for any integer $s,(s\in Z)$ it holds the following formula of inversion to zero*

$$\int_L \lambda^s d\lambda \int_0^1 G_1(x,\xi,\lambda)\psi(\xi)d\xi = 0, \quad 0 < x < 1 \quad . \tag{32}$$

According to [4] we have the following

Lemma 5. *Under constraints 1^0, if $\psi(x)$ the segment [0,1] is Holder continuous with the exponent $q(0 < q \le 1)$, then for $0 < x < 1$ we have the following inversion formulas:*

$$\int_L \lambda^s \tilde{\psi}(x,\lambda)d\lambda = \begin{cases} 0, & as \quad s=0 \\ -\left(\dfrac{\pi}{2}+2\delta\right)\sqrt{-1}\psi(x), & as \quad s=1, \end{cases}$$

where $\tilde{\psi}(x,\lambda)=\int_0^1 P(x,\xi,\lambda)\psi(\xi)d\xi, \quad P(x,\xi,\lambda)$ *is from (78)*

The following theorem on inversion formula follows from lemmas 4 and 5.

Theorem 2. *Under constraints 1^0 and 4^0 , if $\psi(x)$ on the segment [0,1] is Holder continuous with the exponent $q(0<q\le 1)$, then for $0<x<1$ we have the following inversion formulas*

$$\int_L \lambda^s d\lambda \int_0^1 G(x,\xi,\lambda)\psi(\xi)d\xi = \begin{cases} 0, & as \quad s=0 \\ -\left(\dfrac{\pi}{2}+2\delta\right)\sqrt{-1}\psi(x), & as \quad s=1, \end{cases} \tag{33}$$

where $G(x,\xi,\lambda)$ is the Green function of irregular parametric problem (13) (14) .

Stage 4. The solution of the mixed problem.

Using theorem 1 for $\lambda\in R_\delta$ according to the formula (28) from the operation problem (10) (11) we have

$$\tilde{u}(x,t,\lambda)=\delta\left(x,\lambda,-\frac{1}{\lambda^2}e^{-\lambda^2 t}v(t)+\frac{1}{\lambda^2}(f(0)+\tilde{\varphi}_1(t,\lambda)),\tilde{\varphi}_2(t,\lambda)\right)+$$
$$+\int_0^1 G(x,\xi,\lambda)\left[e^{-\lambda^2 t}u(\xi,t)-f(\xi)-\tilde{F}(\xi,t,\lambda)\right]d\xi,\ x\in[0,1],\quad 0\le t\le T.$$

Thus, we established the following

Theorem 3. *Let constraints 1^0, 2^0 and 4^0 be fulfilled. Then , if irregular mixed problem (5)-(7) has a classic solution, then:*

-it is unique,

-this solution is represented by analytic formula

$$u(x,t)=\frac{1}{\pi\sqrt{-1}}\int_L\left\{\frac{1}{\lambda}f(0)e^{\lambda^2 t}\delta(x,\lambda,1,0)+\right.$$
$$+\delta\left(x,\lambda\frac{1}{\lambda}\int_0^t e^{\lambda^2(t-\tau)}\varphi_1(\tau)d\tau,\lambda\int_0^t e^{\lambda^2(t-\tau)}\varphi_2(\tau)d\tau\right)-$$
$$-\lambda e^{\lambda^2 t}\int_0^1 G(x,\xi,\lambda)f(\xi)d\xi-$$
$$\left.-\lambda\int_0^1 G(x,\xi,\lambda)d\xi\int_0^t e^{\lambda^2(t-\tau)}F(\xi,\tau)d\tau\right\}d\lambda \quad 0<x<1,\quad t>0. \tag{34}$$

Remark 6. From derivation of formula (34) it follows that under constraints 1^0, 2^0 and 4^0 if the function $u(x,t)$ defined by formula (34) is not the solution of problem (5)-(7), then this problem has no classic solution.

Remark 7. The positivity of the number $\delta(\delta > 0)$ admits us to get the inequality

$$\left|e^{\lambda^2 t}\right| \le C_T e^{-\omega t|\lambda|^2} \quad for \quad \lambda \in L,\ 0 \le t \le T, \tag{35}$$

where $0 < \omega = \sin 2\delta$.

Inequality (35) admits to take for $t > 0$ the operation of differentiation with respect to *t* and with respect to *x* under the integral sign with respect to L . Imposing definite restrictions on the function $F(x,t), \varphi_1(t), \varphi_2(t), f(x)$ (see [4] it is easy to see that the function $u(x,t)$ determined by the formula (35) in fact is the classic solution of irregular mixed problem (5)-(7).

Model problem 2.

To find the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2}, \quad x \in (0,\infty), t > 0 \tag{36}$$

under the boundary conditions

$$\left.\frac{\partial u}{\partial t}\right|_{x=0} = \varphi(t),\ t > 0 \quad |u(x,t)| \le C_T \exp(M_T x), \quad t \in [0,T],\ as \quad x \to \infty \tag{37}$$

$C_T M_T$ are some constants, in the initial condition

$$u(x,0) = \Phi(x), \quad x \in (0,\infty), \tag{38}$$

where $\varphi(t)\ t \ge 0, \Phi(x), \quad x \ge 0$ are some continuous functions, and $\Phi(x) \le const$ for $x \in [0,\infty)$.

This model problem is among the problems considered by M.L. Rasulov [10]. According to formula (3.5.11), p.152, [10] by Rasulov , the solution of the problem (36)-(38) is

$$u(x,t) = \frac{1}{\pi\sqrt{-1}} \int_L \lambda e^{\lambda^2 t} \left\{ \frac{e^{-\lambda x}}{\lambda^2} \tilde{\phi}(\lambda) + \int_0^\infty G(x,\xi,\lambda)\Phi(\xi)d\xi \right\} d\lambda, \tag{39}$$

where

$$G(x,\xi,\lambda) = \frac{1}{2\lambda}\left[e^{-\lambda|x-\xi|} - e^{-\lambda|x+\xi|}\right], \quad \tilde{\varphi}(\lambda) = \int_0^\infty e^{-\lambda^2 t}\varphi(t)dt. \tag{40}$$

Now, using the finite integral transformation method stated in model problem 1, for solving the problem (36)-(38) we get the following expressions

$$u(x,t)=\frac{1}{\pi\sqrt{-1}}\int_L \lambda e^{\lambda^2 t}\left\{\frac{e^{-\lambda x}}{\lambda^2}\int_0^t e^{-\lambda^2\tau}\phi(\tau)d\tau+\frac{\Phi(0)}{\lambda^2}e^{-\lambda x}+\int_0^\infty G(x,\xi,\lambda)\Phi(\xi)d\xi\right\}d\lambda, \tag{41}$$

where G is from (40).

Thus, the solution of problem (36)-(38) is represented by different formulas (39) and (41). Comparing (39) and (41) we see that the second summand in braces (41), (39) do not exist, further in (39) in the first summand the integral $\int_0^t e^{-\lambda^2\tau}\varphi(\tau)d\tau$ is replaced by $\int_0^\infty e^{-\lambda^2\tau}\varphi(\tau)d\tau$. This replacement, in our opinion, is not successful.

In the problem (36)-(38) we assume

$$\varphi(t)\equiv 0,\quad \Phi(x)\equiv 1. \tag{42}$$

Then from (39) we have

$$u(x,t)=\frac{2}{\sqrt{\pi}}\int_0^{x/(2\sqrt{t})}\exp(-\xi^2)d\xi, \tag{43}$$

and from (41) we get

$$u(x,t)\equiv 1 \tag{44}$$

On the other hand, by theorem 12, p 152 [10], the solution of the problem (36)-(38), (42) is unique, on the other hand, for the solution $u(x,t)$ of problem (36)-(38), (42) M.L. Rasulov obtains the expression (43), though in fact the classic solution of the problem (36)-(38), (42) is $u(x,t)\equiv 1$ from (44).

1.Statement of the problem

Find then classic solution of the equation

$$\frac{\partial u}{\partial t}-Z\left(x,\frac{\partial}{\partial x}\right)u-f(x,t)=v(x),\quad 0<x<1,\quad 0<t<T, \tag{45}$$

$$Z\left(x,\frac{\partial}{\partial x}\right)\equiv a(x)\frac{\partial^2}{\partial x^2}+b(x)\frac{\partial}{\partial x}+c(x),$$

satisfying the integro-differential "boundary" conditions

$$V_i(D_t,D_x)u\equiv\sum_{n=0}^{1}\sum_{j=0}^{1}\Big\{\alpha_{jn}^{(i)}D_t^jD_x^n\,u(x,t)/_{x=0}+\beta_{jn}^{(i)}D_t^jD_x^n\,u(x,t)/_{x=1}+$$

$$\left.+\int_0^1\gamma_{j_n}^{(i)}(x)D_t^jD_x^n u(x,t)dx\right\}=\varphi_i(t),\quad 0<t<T,\quad i=1,2 \tag{46}$$

the initial condition

$$u(x,t)\big|_{t=0}=f_1(x),\quad 0<x<1, \tag{47}$$

and the finite condition

$$u(x,t)\big|_{t=T} = f_2(x), \quad 0 < x < 1, \tag{48}$$

where $u \equiv u(x,t)$ is the sought-for continuous classic solution; $v(x)$ is the sought-for continuous control, $a(x), b(x), c(x), f(x,t), \gamma_{jn}^{(i)}(x), \varphi_i(t), f_i(x)$, are known functions ; $T(T>0)$, $\alpha_{jn}^{(i)}, \beta_{jn}^{(i)}$ are known numbers.

1^0. Let equations (45) be parabolic in I.G. Petrovsky sense, i.e. let $\operatorname{Re} a(x) \geq \delta_0, \quad x \in [0,1]$, where $\delta_0 (\delta_0 > 0)$ is some number.

2^0. Let $a(x) \in C^{m+2}([0,1]), b(x) \in C^{m+1}([0,1]), c(x) \in C^m([0,1])$.

3^0. Let $\gamma_{jn}^{(i)} \in C^q([0,1]), i = 1,2;\ j = 0,1;\ n = 0,1.$

In conditions 2^0 and 3^0 , $m(m \geq 0)$, $q(q \geq 0)$, are some integers (see remark 2.2).

4^0. Let

$f_i(x) \in C^1([0,1]), \varphi_i(t) \in C([0,T]), (i=1,2), f(x,t) \in C([0,1]\times[0,T]).$

From constraint 1^0 it follows that

$$-\frac{\pi}{2} + 4\delta \leq \arg a(x) \leq \frac{\pi}{2} - 4\delta,\ x \in [0,1], \tag{49}$$

where $\delta\left(0 < \delta < \dfrac{\pi}{8}\right)$ is some positive number.

Definition 1.1. It is said that the function $u \equiv u(x,t)$ is a classic solution of problem (45) -(48) if

a_1: the function $u(x,t)$ is continuous for $0 \leq x \leq 1, 0 \leq t \leq T.$

a_2: the function $u(x,t)$ for $0 \leq x \leq 1, 0 < t < T$ has a continuous derivative of the form

$$\frac{\partial u(x,t)}{\partial x}, \frac{\partial^2 u(x,t)}{\partial x^2};$$

a_3: the function $u(x,t)$ for $0 < x < 1, 0 < t < T$ has continuous derivatives of the form $\dfrac{\partial u(x,t)}{\partial t}$;

a_4: if $\alpha_{10}^{(i)} \neq 0$, then $\dfrac{\partial u(x,t)}{\partial t} \in C([0,\alpha]\times(0,T))$ where $\alpha(0<\alpha<1)$ is some number;

if $\beta_{10}^{(i)} \neq 0$, then $\dfrac{\partial u(x,t)}{\partial t} \in C([1-\alpha,1]\times(0,T));$

if $\alpha_{11}^{(i)} \neq 0$, then $\frac{\partial^2 u(x,t)}{\partial x \partial t} \in C([0,\alpha]\times(0,T)), \frac{\partial u(x,t)}{\partial x} \in C([0,\alpha]\times[0,T))$;

if $\beta_{11}^{(i)} \neq 0$, then

$\frac{\partial^2 u(x,t)}{\partial x \partial t} \in C([1-\alpha,1]\times(0,T)), \frac{\partial u(x,t)}{\partial x} \in C([1-\alpha,1]\times[0,T))$;

if $\gamma_{10}^{(i)}(x)$ is not identically equal to zero,then $\frac{\partial u(x,t)}{\partial t} \in C([0,1]\times(0,T))$,

if $\gamma_{11}^{(i)}(x)$ is not identically equal to zero, then $\frac{\partial^2 u(x,t)}{\partial x \partial t} \in C([0,1]\times(0,T))$,

$\frac{\partial u(x,t)}{\partial x} \in C([0,1]\times[0,T))$;

a_5: the function $u(x,t)$ satisfies equalities (45)-(48) in the ordinary sense.

2.Parametric problem.

For solving problem (45)-(48) as first we solve the following parametric problem

$$\left(Z\left(x,\frac{d}{dx}\right)-\lambda^2\right)y=\psi(x), \quad x\in(0,1), \tag{50}$$

$$V_i\left(\lambda^2,\frac{d}{dx}\right)y=\gamma_i, \quad i=1,2, \tag{51}$$

where $\psi(x)\in C([0,1]), \gamma_i$ are some numbers, λ -is a complex parameter.

Here and is the sequel, unless otherwise stipulated, we assume

$$\lambda \in R_\delta \equiv \left\{\lambda : |\lambda| \geq R, \quad |\arg\lambda| \leq \frac{\pi}{4}+\delta\right\},$$

where δ is from (49), R is a rather large positive number.

Denote by $\theta_i(x), (i=1,2)$ the roots of the characteristical equation [1, 9,10,11], covresponding to (50), i.e. let

$$-\theta_2(x)=\theta_1(x)=-|a(x)|^{\frac{1}{2}}\exp\left(-\sqrt{-1}\,\frac{\arg a(x)}{2}\right).$$

According to [1,9,10,11], [4], under constraints 2^0 and $a(x)\neq 0$, $x\in[0,1]$ there exist the functions

$$g_{is}(x)\in C^2([0,1]), \ s=\overline{0,m}, \quad g_{i0}(x)\neq 0, \ x\in[0,1],$$

that for the functions

$$y_{im}(x,\lambda)\equiv \exp\left(\lambda\int_0^x \theta_i(\xi)d\xi\right)\left[g_{i0}(x)+\frac{1}{\lambda}g_{i1}(x)+...+\frac{1}{\lambda^m}g_{im}(x)\right],$$

it holds

$$\left(Z\left(x,\frac{d}{dx}\right)-\lambda^2\right)y_{im}(x,\lambda)=\exp\left(\lambda\int_0^x \theta_i(\xi)d\xi\right)\frac{E_{im}(x)}{\lambda^m},\quad x\in[0,1], i=1,2$$

$\lambda(\lambda\neq 0)$ is any (complex) number, $C([0,1])\ni E_{im}(x)$ is some function.

In [4] in domain R_δ the fundamental solution $P(x,\xi,\lambda)$ of equation (50) is constructed in the form

$$P(x,\xi,\lambda)=P_0(x,\xi,\lambda)+\int_0^1 P_0(x,\eta,\lambda)h(\eta,\xi,\lambda)d\eta, \tag{52}$$

($h(x,\xi,\lambda)$ -is a sought -for kernel), where

and using fundamental solution, under constraints 1^0 and 2^0 in [4] it is proved that a homogeneous equation corresponding to (50) has the system of fundamental particular solutions $y_i(x,\lambda)$, $(i=1,2)$ that together with first order derivatives are represented by the asymptotic formulas

$$\frac{d^s}{dx^s}y_1(x,\lambda)=\frac{d^s}{dx^s}y_{1m}(x,\lambda)+E_{1m}^{(s)}(x,\lambda),$$

$$\frac{d^s}{dx^s}y_2(x,\lambda)=\exp\left(-\lambda\int_0^1 \theta_2(\xi)d\xi\right)\frac{d^s}{dx^s}y_{2m}(x,\lambda)+E_{2m}^{(s)}(x,\lambda),\ s=0,1; \tag{53}$$

where $E_{im}^{(s)}(x,\lambda)$ are some continuous functions with respect to $x\in[0,1]$ and analytic with respect to $\lambda\in R_\delta$ and the following estimations hold:

$$\left|E_{1m}^{(s)}(x,\lambda)\right|\leq\frac{C}{|\lambda|^{m-s}}\exp(-\varepsilon|\lambda|x),$$

$$\left|E_{2m}^{(s)}(x,\lambda)\right|\leq\frac{C}{|\lambda|^{m-s}}\exp(-\varepsilon|\lambda|(1-x)),\ \left|y_1(x,\lambda)\right|\leq C\exp(-\varepsilon|\lambda|x),$$

$$\left|y_2(x,\lambda)\right|\leq C\exp(-\varepsilon|\lambda|(1-x)). \tag{54}$$

Here and in the sequel, by C and ε we denote different positive numbers (concrese values of which are not important) independent of $x\in[0,1]$ and $\lambda\in R_\delta$

Remark 2.1. In the classic papers [9, 10,11], the methods used by them compelled them to know asymptotics of the system of fundamental particular solutions of a homogeneous equation with respect to λ along the direction of λ - plane and because of that, on the roots of characteristical equation they imposed the constraints:

(A): the arguments $\theta_i(x)$ and arguments of their differences $\theta_i(x) - \theta_j(x)$ are independent of x.

For equation (50) constraints (A) reduce to the assumption

(B): $a(x) = \alpha P(x)$, where $P(x) > 0,\ \alpha - const$ for $x \in [0,1]$.

Unlike classic papers [9, 11], in the present paper (and in [4]) constraints (A) are not imposed on the roots of the characteristical equation and therefore fulfilment of constraint (B) is not supposed.

Expanding the determinant (determinator of the Green function) we have

$$\Delta(\lambda) = Q(\lambda) + E(\lambda), \tag{55}$$

where

$$Q(\lambda) = \alpha_6 \lambda^6 + \alpha_5 \lambda^5 + ... + \alpha_{6-K} \frac{1}{\lambda^{K-6}},\ E(\lambda) = O\left(\frac{1}{\lambda^{K-5}}\right), \tag{56}$$

$K (K \geq 0)$ is some integer.

Note that the number K contained in (56), may be taken rather large (i.e. for $\Delta(\lambda)$ one can obtain sufficiently exact asymptotic representations) if the numbers m (from 2^0, and q (from 3^0) are rather large.

Definition 2.1. It is said that boundary conditions (46)(or (51)) well-defined right if at least one the numbers $\alpha_6, \alpha_5, ..., \alpha_{6-K}$, (from (56)) is non-zero.

If boundary conditions of the spectral problem are regular in Brikhoff - Tamarkin - Naimark - Rasulov’s sense, then they are well-defined in the sense of our definition. But the inverse statement is not true.

Show this on the following examples

$$y'' - \lambda^2 y = \psi(x), \quad x \in (0,1), \tag{57}$$

$$y(0) - 2y(1) = 0, \quad y(0) + y'(0) + 2y'(1) = 0 \tag{58}$$

If we take system of fundamental particular solutiouns of a homogeneous equation corresponding to (57), in the form $y_1(x,\lambda) = \exp(-\lambda x), y_2(x,\lambda) = \exp(\lambda x)$, then the denomination of the Green functions of problem (57)-(58) will be

$$\Delta(\lambda) = 2e^{\lambda} - 6\lambda - 2e^{-\lambda}.$$

Hence it is seen that boundary conditions (58) for equation (57) are regular in the sense of Brikhoff, Tamarkin, Naimak, Rasulov. It we take the system of fundamental particular solutions of homogeneous equations corresponding to (57) in the form (53), i.e., $y_1(x,\lambda) = \exp(-\lambda x),\ y_2(x,\lambda) = \exp(-\lambda(1-x))$,

then the denominator of the Green function of problem (57)-(58) will be (55) (the Green function itself of the parametric problem is independent on the choice of the system of fundamental particular solutions), where $Q(\lambda) = 2, E(\lambda) = -6\lambda e^{-\lambda} -$

$2e^{-2\lambda} = O\left(\frac{1}{\lambda^S}\right)$, $\lambda \in R_\delta$ (S is any natural number)that indicates well-poseduess of boundary conditions (58) in the sense of definition 2.1.

Let condition (51) contain only the integrals, i.e. for example, let

$$V_i(y) \equiv \int_0^1 K_i(x) y(x,\lambda) dx = \gamma_i, \quad i = 1,2, \tag{59}$$

5^0. Let $K_i(x) \in C^1([0,1])$ $\alpha \equiv K_1(0)K_2(1) - K_1(1)K_2(0) \neq 0$.

Then we have

$$V_i(y_1) \equiv \int_0^1 K_i(x)\exp(-\lambda x)dx = \frac{1}{\lambda}K_i(0) - \frac{1}{\lambda}e^{-\lambda}K_i(1) +$$

$$+\frac{1}{\lambda}\int_0^1 K_i^{'}(x)\exp(-\lambda x)dx = \frac{1}{\lambda}K_i(0) + O\left(\frac{1}{\lambda^2}\right);$$

$$V_i(y_2) \equiv \int_0^1 K_i(x)\exp(-\lambda(1-x))dx = \frac{1}{\lambda}K_i(1) - \frac{1}{\lambda}e^{-\lambda}K_i(0) -$$

$$-\frac{1}{\lambda}\int_0^1 K_i^{'}(x)\exp(-\lambda(1-x))dx = \frac{1}{\lambda}K_i(1) + O\left(\frac{1}{\lambda^2}\right);$$

therefore from (56) we have

$$Q(\lambda) = \frac{\alpha}{\lambda^2}, \; E(\lambda) = O\left(\frac{1}{\lambda^2}\right).$$

Consequently under constraints 5^0 , for parametric problem (57),(59) "boundary conditions" (59) are defined by our definition.

6^0. Let boundary conditions (46) (and (51)) be well defined.

Further, let among the non-zero numbers $\alpha_6, \alpha_5, ..., \alpha_{6-K}$ (from (56)), with the greatest index there exist α_{6-M} .

Remark 2.3. It is approriate to take in condition 2^0 the number m and in condition 3^0 the number q so least at which for $Q(\lambda)$ (from (56)) it holds

$$Q(\lambda) = \frac{1}{\lambda^{M-6}}\alpha_{6-M}. \tag{60}$$

Using (60), for rather large R, from (55) we have

$$|\Delta(\lambda)| \geq \frac{1}{2}|\alpha_{6-M}|\frac{1}{|\lambda|^{M-6}} \tag{61}$$

Inequality (61) shows that $\Delta(\lambda) \neq 0$ for $\lambda \in R_\delta$. Therefore, according to [1,9,10, 11] and [4], it is hold following

Lemma 2.1. *Let constraints* 1^0 ,2^0 *and* 6^0 *be fulfilled. Then, for* $\lambda \in R_\delta$ *and* $\psi(x) \in C([0,1])$ *parametric problem* (50), (51):

i) *has a unique solution* $y(x,\lambda)$, ,

ii) *this solution is an analytic function with respect to* $\lambda \in R_\delta$,

iii) *the solution* $y(x,\lambda)$, *is represented by the formula*

$$y(x,\lambda) = \int_0^1 G(x,\xi,\lambda)\psi(\xi)d\xi + \delta(x,\lambda,\gamma_1,\gamma_2), \tag{62}$$

Using (54) and (61) we get the estimations

$$|G_1(x,\xi,\lambda)| \le C|\lambda|^N [\exp(-\varepsilon|\lambda|x) + \exp(-\varepsilon|\lambda|(1-x))],$$

$$|\delta(x,\lambda,\gamma_1,\gamma_2)| \le C|\lambda|^N (|\gamma_1| + |\gamma_2|)[\exp(-\varepsilon|\lambda|x) +$$

$$+\exp(-\varepsilon|\lambda|(1-x))],\ x,\xi \in [0,1],\ \lambda \in R_\delta \tag{63}$$

where *N* is some (integer) number, *C* is a constant, independent of $x,\xi \in [0,1]$ $\lambda \in R_\delta$ and γ_1, γ_2 .

In [4] the following theorem on inversion formula was proved by the fundamental solution $P(x,\xi,\lambda)$ from (52).

Theorem 2.1. *Let constraints* 1^0 *and* 2^0 *be fulfilled. Then for any absolutely continuous functions* $\psi(x)$ *on* [0,1] *it holds the following inversion formula*

$$-\frac{1}{\left(2\delta + \frac{\pi}{2}\right)\sqrt{-1}} \int_L \lambda^S d\lambda \int_0^1 P(x,\xi,\lambda)\psi(x)d\xi = \begin{cases} \psi(x), & \text{for } S = 1,\ 0 < x < 1, \\ 0, & \text{for } S = 0,\ -1, -2, \ldots, \end{cases} \tag{64}$$

where L is an infinite smooth line in R_δ *, whose rather distant part coincides with continuation of the rays* $\arg\lambda = \pm\left(\frac{\pi}{4} + \delta\right)$, *moreover in* (64) *the integral along the lines L is understood in the seuse of prinspal value.*

Taking into account estimations (63), by the method stated in [4], we easily prove the following

Theorem 2.2. *Let constraints* 1^0,2^0, 3^0 *and* 6^0 *be fulfilled. Then for any functions* $\psi(x) \in C([0,1])$ *and arbitrary numbers* γ_1 *and* γ_2 *it holds the following invertion formula*

$$\int_L \lambda^S d\lambda \int_0^1 G_1(x,\xi,\lambda)\psi(\xi)d\xi = 0, \quad 0 < x < 1 \tag{65}$$

$$\int_L \lambda^S \delta(x,\lambda,\gamma_1,\gamma_2)d\lambda = 0, \quad 0 < x < 1, \tag{66}$$

where S *-is any integer, L -is from theorem 2.1.*

From Theorems 2.1 and 2.2 we obtain the following

Theorem 2.3. *Let constraints* $1^0, 2^0, 3^0$ *and* 6^0 *be fulfilled. Then for any absolutely continuous functions* $\psi(x)$ *on* $[0,1]$ *it holds the following inversion formula:*

$$-\frac{1}{\left(2\delta+\frac{\pi}{2}\right)\sqrt{-1}}\int_L \lambda^S d\lambda \int_0^1 G(x,\xi,\lambda)\psi(\xi)d\xi = \begin{cases}\psi(x), & for\ S=1,\ 0<x<1,\\ 0, & for\ S=0,-1,-2,...,\end{cases} \tag{67}$$

where L is from theorem 2.1.

In [4] the following theorem on inversion formula is proved.

Theorem 2.4. *If the function* $\varphi(t)$ *is continuous for* $t \ge 0$, *then it holds the following inversion formula*

$$-\frac{1}{\left(2\delta-\frac{\pi}{2}\right)\sqrt{-1}}\int_L \lambda\tilde{\phi}_0(t,\lambda)d\lambda = \varphi(t), \quad (t>0), \tag{68}$$

L is from theorem 2.1 , $\tilde{\phi}_0(t,\lambda) \equiv \int_0^t \exp\left(\lambda^2(t-\tau)\right)\varphi(\tau)d\tau$.

3.Solution of the mixed problem.

Let the sought-for continuous control $v(x)$ be a priori known and problem (45)-(47) have the classic solution $u \equiv u(x,t)$. Applying to (45)-(47) the finite integral transformation [4] , we get

Theorem 3.1. *Let constraints* $1^0, 2^0, 3^0, 4^0$ *and* 6^0 *be fulfilled. Then, if problem (45)-(47) has a unique solution, this is a unique solution and is represented by formula*

$$u(x,t) = \frac{1}{\pi\sqrt{-1}}\int_L \left\{\frac{1}{\lambda}(1-\exp(\lambda^2 t))\int_0^1 G(x,\xi,\lambda)v(\xi)d\xi + \lambda F(x,t,\lambda)\right\}d\lambda,$$

$$0 < x < 1, \quad 0 < t \le T. \tag{69}$$

$$F(x,t,\lambda) = \delta(x,\lambda,\psi_{10}(t,\lambda),\psi_{20}(t,\lambda)) - \int_0^1 G(x,\xi,\lambda)\left[\exp(\lambda^2 t)f_1(\xi) + \tilde{f}_0(\xi,t,\lambda)\right]d\xi$$

Substituting (69) in finite condition (48) for determining the unknown control we get Fredholm's second order integral equation ($0 < x < 1$)

$$f_2(x) = \frac{1}{\pi\sqrt{-1}}\int_L \left\{\frac{1}{\lambda}(1-\exp(\lambda^2 T))\int_0^1 G(x,\xi,\lambda)v(\xi)d\xi + \lambda F(x,T,\lambda)\right\}d\lambda,. \tag{70}$$

Because of restriction on the paper's volume, we don't give investigation of the solution of equation (70).

Solving equation (70), we find the unknown control and according to (67) we have

$$\int_L \frac{1}{\lambda} d\lambda \int_0^1 G(x,\xi,\lambda) v(\xi) d\xi = 0, \quad 0 < x < 1. \tag{71}$$

Taking into account (71) in (69), we get ($0 < x < 1, \quad 0 < t \le T$)

$$u(x,t) = \frac{1}{\pi\sqrt{-1}} \int_L \left\{ -\frac{1}{\lambda} \exp(\lambda^2 t) \int_0^1 G(x,\xi,\lambda) v(\xi) d\xi + \lambda F(x,t,\lambda) \right\} d\lambda \tag{72}$$

Improsing restraints on rather smooth ness of the functions contained in 3^0 and 4^0 and using the inequality

$$\left|\exp(\lambda^2 t)\right| \le C \exp\left(-|\lambda|^2 t \sin 2\delta\right), \quad \lambda \in L. \tag{73}$$

by the method stated in [4], it is easily shown that the function $u(x,t)$, , determined by formula (73), is the solution of problem (45)-(48).

Remark 3.2. In (72) when substituting $z = \lambda^2$ the obtained image of the L line from the Laplace straight line and advantage of the differs L from the Laplace straight line is that according to inequality (73), in (72) the integrand factor $(\lambda^2 t)$ for $t > 0$, $\lambda \in L$, $|\lambda| \to \infty$, decreases exponent tially, and this allows easily to justify convergence of the integral in unbounded lines L .

4. Model problem.

Problem statement. Find the solution of the equation

$$\frac{\partial u}{\partial t} = \frac{\partial^2 u}{\partial x^2} + v(x), \quad x \in (0,1), 0 < t < T, \tag{74}$$

with the unknown control $v(x)$, satisfying the boundary conditions

$$u(x,t)\big|_{x=0} = 0, \; u(x,t)\big|_{x=1} = 0 \quad 0 < t < T, \tag{75}$$

initial condition

$$u(x,t)\big|_{t=0} = f_1(x) \quad 0 < x < 1, \tag{76}$$

and finite condition

$$u(x,t)\big|_{t=T} = f_2(x) \quad 0 < x < 1. \tag{77}$$

Let $C_0^l([0,1]) = \{\varphi(x) : \varphi(x) \in C^l([0,1]), \varphi^{(k)}(0) = \varphi^{(k)}(1) = 0,$ for $k = 0,\ldots,l-1\}$.

All constraints of theorem 3.1 are fulfilled for problem (74) - (77) , and according to formula (69), we have

Theorem 4.1. *Let* $f_1(x) \in C_0^2([0,1]), \;\; f_2(x) \in C_0^4([0,1]).$ *Then mixed problem(74)-(77) with a control has a unique classical solution and this solution is represented by formula*

$$u(x,t) = 2\sum_{k=1}^{\infty} \exp(-k^2\pi^2 t)\sin k\pi x \int_0^1 f_1(\xi)\sin k\pi\xi d\xi + 2\sum_{k=1}^{\infty} \frac{1}{k^2\pi^2} \times$$

$$\times (1 - \exp(-k^2\pi^2 t))\sin k\pi x \int_0^1 v(\xi)\sin k\pi\xi d\xi, \ 0 < x < 1, \quad 0 < t \le T. \tag{78}$$

$$\int_0^1 v(\xi)\sin k\pi\xi d\xi = \frac{k^2\pi^2}{1 - \exp(-k^2\pi^2 T)} \times$$

$$\times \left[\int_0^1 f_2(\xi)\sin k\pi\xi d\xi - \exp(-k^2\pi^2 T) \int_0^1 f_1(\xi)\sin k\pi\xi d\xi \right]. \tag{79}$$

$$v(x) = 2\sum_{k=1}^{\infty} \sin k\pi x \int_0^1 v(\xi)\sin k\pi\xi d\xi. \tag{80}$$

where the sought-for control $v(x)$ *is found by formula (80), its Fourier coefficients* [3] *by formula (79).*

The finite integral transformation method used here was suggested by us in [4] and was successfuley used in [5-8] and in others.

References

1. Brikhoff G.D., On the asymptotic character of the solutions of certain linear differential equations containing a parameter. *Trans. AM Math. Soc*., 9 (1908), 219-232
2. Cauchu A.L., Me'moire'sur l'application du calcul des residus a'la solution des problems de physique mathematique. Paris, 7 (1827), 1-56
3. Fichtenholts G.M. *Course of differential intergral calculus* vol. III, Moscow,"Nauka",(1970), 656 p.
4. Gasymov Elmaga *Finite integral transformation method*. Baku, "Elm", (2009), 434 p.
5. Gasymov E.A. *Mixed problem on conjugation of parabolic systems of different order with nonlocal boundary conditions*. Diff. Uravn., **26**(8) (1990), 1364-1374.
6. Gasymov E.A., Application of finite integral transformation method to the solution of mixed problem with integral differential conditions for one nonclassical equation *Diff. Uravn*., **47**(3),(2011), 322-334.
7. Gasymov E.A., Studying mixed problems in conjugation of hyperbolic systems of different orders. *Zhournal vychislitelnoy matematiki i matematichiskoy fiziki*, **52**(8) (2012), 1472-1481.
8. Gasymov E.A., Guseynova A.O., Gasanova U.N., Application of the generalized method of separation of variables to the solution of the displaced

problem with irregular boundary conditions *Zhournal vychislitelnoy matematiki i matematichiskoy fiziki*, **56**(7) (2016), 1335-1339.
9. Naimark M.A., *Linear differential operators*. M. Nauka, (1969).
10. Rasulov M.L., *Application of the contour integral method* . M. Nauka, (1975).
11. Tamarkin Ya. D., *On some general problems of theory of ordinary linear differential equations on expansions of arbitrary functions in series.* Petrograd, (1917).

Применение метода конечного интегрального проеобразования к решению смещанных задач для параболических уравнений с управлением

Э.А.Гасымов
gasymov-elmagha@rambler.ru

РЕЗЮМЕ

Одним из методов решения смещанных задач является классический метод разделения переменных (метод Фурье). Когда граничных условия смещанной задачи нерегульярны, то вообще говоря, этот метод неприменимы. В настоящей работе применяется метод конечного интегрального преобразования к решению смещанных задач для параболических уравнений с управлением и с нерегульярными граничными условиями. Получено аналитическое представление решения рассматриваемой смещанной задачи.

Ключевые слова: классическое решение, метод конечного иетегрального преобразования, уравнения с управлением, нерегульярное граничное условие